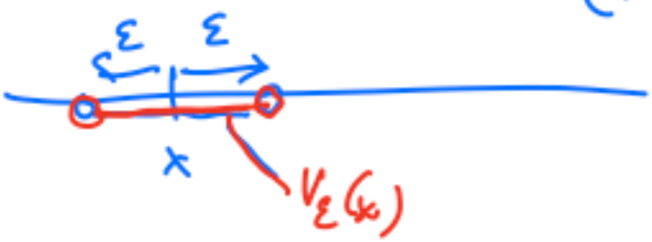
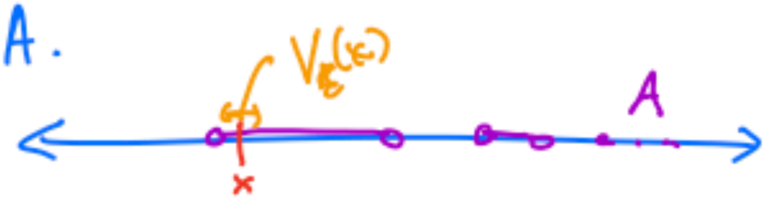
Topology – study of open & closed sets.

Given $x \in \mathbb{R}$, the <u>ε-neighborhood</u> of x in $\mathbb{R}$ is

$$V_\varepsilon(x) = \{ y : |y-x| < \varepsilon \}$$
$$= (x-\varepsilon, x+\varepsilon)$$



A set $A \subseteq \mathbb{R}$ is called <u>open</u> if $\forall x \in A$, $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq A$.



<u>Example:</u> open intervals are open sets. Also, unions of open intervals are open sets.

(Intuitively, a set is open if it doesn't contain its boundary.)

Lemma: (Arbitrary unions of open sets are open). Suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of open sets in $\mathbb{R}$. ($\mathcal{A}$: index set.)

Then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is also open in $\mathbb{R}$.

Pf. With notation above, $\forall x \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, x must be in at least one of the sets U_α, so since U_α is open, $\exists V_\varepsilon(x)$ (nbhd of x) s.t. $V_\varepsilon(x) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Thus $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is open. $\boxed{\emptyset}$

Lemma. $\emptyset$ is an open set.

(vacuously satisfies the property.)

Lemma. If $U_1, \ldots, U_n$ are open sets in $\mathbb{R}$, then $\bigcap_{j=1}^{n} U_j$ is also open.

(Finite intersections of open sets are open)

Pf. With notation as above, suppose $x \in \bigcap_{j=1}^{n} U_j$. Since U_j is open and $x \in U_j$ $\forall j$,

$\exists \varepsilon_j > 0$ s.t. $V_{\varepsilon_j}(x) \subseteq U_j$.

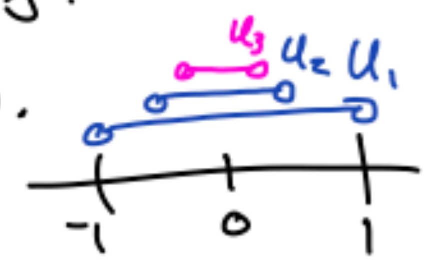
Then let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$.

Then $V_\varepsilon(x) \subseteq V_{\varepsilon_j}(x) \subseteq U_j \quad \forall j$,

so $V_\varepsilon(x) \subseteq \bigcap_{j=1}^{n} U_j$. ☒

<u>Remark</u>: The proof above fails if there is an ∞ # of sets. eg.

Let $U_j = \left(-\frac{1}{j}, \frac{1}{j}\right)$.

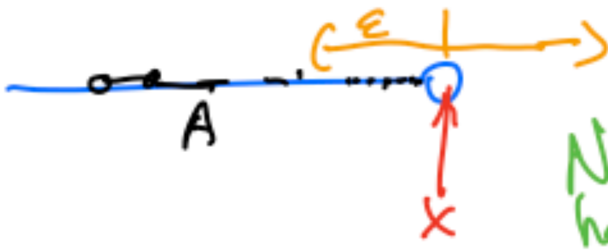


Then $\bigcap_{j=1}^{\infty} U_j = \{0\}$ ← a single pt.

Not open, because $\forall \varepsilon > 0$, $\exists\, y = \frac{\varepsilon}{2} \in V_\varepsilon(0)$, where $y \notin \{0\}$.

Closed sets.

Defn If $A \subseteq \mathbb{R}$, we say that a point $x \in \mathbb{R}$ is a __limit point__ of A if $\forall \varepsilon > 0$, $V_\varepsilon(x)$ contains a point $y \in A$ s.t. $y \neq x$.



Note: x does not have to be in the set at all.

__eg:__ If $(a,b) \subseteq \mathbb{R}$ is an open interval with $a < b$. Then

(1) Every point of (a,b) is a limit point of (a,b).

(2) Also, a & b are limit pts of (a,b).

__Pf.__ (1) Suppose $x \in (a,b)$, so that

$$a < x < b.$$

$\forall \varepsilon > 0$, $V_\varepsilon(x) \cap (a,b)$ is also an open interval containing x, which must contain points besides x, which are therefore in (a,b). ✓

(2). Observe that $\forall \varepsilon > 0$, $V_\varepsilon(a) \cap (a,b)$
$= (a, \min\{b, a+\varepsilon\})$

↑ This is a nonempty interval inside (a,b), so it contains points of (a,b).

Thus a is a limit pt of (a,b).

Similarly, b is a limit pt of (a,b). $\square$

Eg: Let $C = \{ \frac{1}{n} : n \in \mathbb{N} \}$.

0 is the only limit point of C.

Defn – A point of a set A that is not a limit point of A is called an isolated pt. (i.e. x is isolated $\Rightarrow$ $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) \cap A = \{x\}$.)

We denote the set of limit points of A by L_A. (Note: it's not always true that $L_A \subseteq A$.)

Lemma. (Sequential Defn of Limit Pt.)

A number $x \in \mathbb{R}$ is a limit point of a set A if $\exists$ sequence (a_n) of points in $A \setminus \{x\}$ s.t. $\lim a_n = x$.

set minus

for two sets C & D, $C \setminus D = C \cap D^c$

$= \{x : x \in C \text{ and } x \notin D\}$.

Defn A set $A \subseteq \mathbb{R}$ is closed if $L_A \subseteq A$. (ie if A contains all of its limit points.)

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eg. $\emptyset$ is closed.

Finite sets are closed.

$[a,b]$ is closed.

$A = \{\frac{1}{n} : n \in \mathbb{N}\}$ not closed because $\{0\} \in L_A$.

But $\{0\} \cup A$ is closed.

Lemma (Arbitrary intersections of closed sets are closed).

Let $\{F_\alpha\}_{\alpha \in \mathcal{Q}}$ be a collection
index set

of closed sets. Then $\bigcap_{\alpha \in \mathcal{Q}} F_\alpha$ is also closed.

Lemma (Finite unions of closed sets are closed). If $F_1, \dots, F_K$ are closed sets, then $\bigcup_{j=1}^{K} F_j$ is also closed.

Ex A single point $\{2.7\}$ is a closed set (no limit pts)

$\{1, -1, 3, 7, 11.8\}$ is also closed (no limit pts),

$\forall n \in \mathbb{N}$. $\{\frac{1}{n}\}$ is closed, but

$\bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$ is not closed. (0 is a limit pt.)

Lemma If C is a closed set in $\mathbb{R}$, and if (a_n) is a Cauchy seq. (i.e. convergent sequence) of points in C, then

$$L = \lim a_n \in C.$$

You can make this into a definition of a closed set:

A set C is closed $\iff$ every Cauchy seq. of points in C converges to a point in C.

Big Theorem: If $U \subseteq \mathbb{R}$ is open, then $U^c = \{y \in \mathbb{R} : y \in U\}$ is closed.
Conversely, if $F \subseteq \mathbb{R}$ is closed, then F^c is open.

Warning: • There are lots of sets that are neither closed nor open.

– $[0, 3)$, $\{\frac{1}{n} : n \in \mathbb{N}\}$, etc.

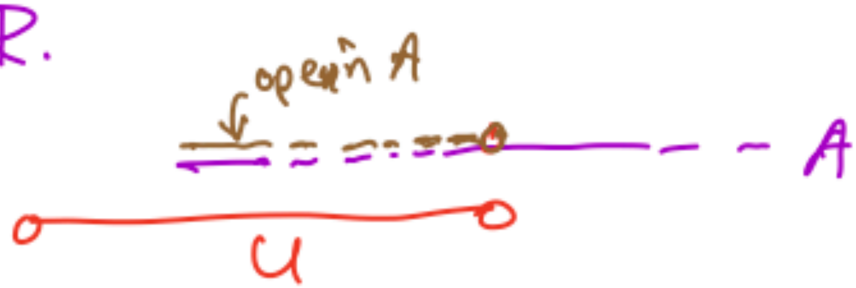
- There are examples of sets that are both closed and open.
  - $\emptyset$
  - $\mathbb{R}$

  } These are the only two sets that are both closed & open.

Subspace Topology.

Suppose $X \subseteq \mathbb{R}$ is nonempty.

We say that a set A in X is open in X if $A = U \cap X$, where U is open in $\mathbb{R}$.



Similarly, a set B in X is closed in X if $B = F \cap X$, where F is closed in $\mathbb{R}$ $\iff$ B = complement of an open set in X.

Thm Given a set $S \subseteq \mathbb{R}$, let $\overline{S} = S \cup L_S$. Then $\overline{S}$ is closed.

$\overline{S}$ is called the closure of S

Also: $\overline{S}$ is the smallest closed set that contains S. ie. If F is closed and $F \supseteq S$, then also $F \supseteq \overline{S}$.

Another way to consider $\overline{S}$:

$$\overline{S} = \bigcap_{\substack{F \text{ is closed} \\ \text{and } S \subseteq F}} F$$

Review of de Morgan's Laws. If $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of subsets of $\mathbb{R}$, then $\left(\bigcup_{\alpha\in\mathcal{A}} S_\alpha\right)^c = \bigcap_{\alpha\in\mathcal{A}} (S_\alpha^c)$

and $\left(\bigcap_{\alpha\in\mathcal{A}} S_\alpha\right)^c = \bigcup_{\alpha\in\mathcal{A}} (S_\alpha^c)$.

From the above, we can use the fact that arbitrary unions of open sets are open to prove that arbitrary intersections of closed sets are closed.

Exercise 3.2.2. Let

$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\} \quad \text{and} \quad B = \{ x \in \mathbf{Q} : 0 < x < 1 \}.$$

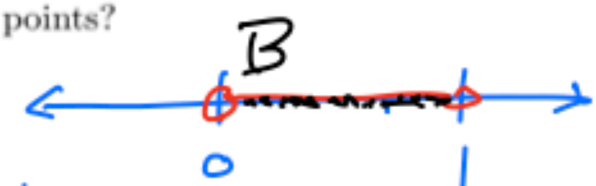
Answer the following questions for each set:

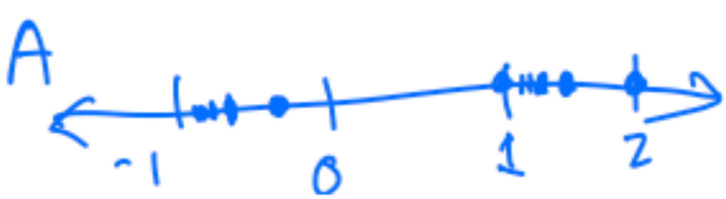
(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.





(a) $L_A = \{-1, 1\}$ $L_B = [0, 1]$

every # in here is a limit of a sequence in B.

(b)